A SPECIAL CASE OF THE EXISTENCE OF SMALL PERIODIC MOTIONS OF TWO PENDULI, SUBJECTED TO UNIFORM ROTATION

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PMM Vol.22, No.1, 1958, pp.139-142

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(Received 28 November 1955)

The author has previously considered [2] the following system of penduli. A Cartesian, right-handed coordinate system $Oxyz$ rotates with a constant angular velocity ω about the z-axis, directed vertically downward. At a point A_1, lying on the horizontal y-axis at a distance R from the origin O, a heavy rod A_1A_2 of length $2a$ and mass m, is suspended such that OA_1 will be the oscillation axis. Attached to the end A_2 of this rod is another heavy rod A_2A_3 of the same length and mass, with its oscillation axis parallel to OA_1 etc. Finally, at the end A_n of the next to the last rod, the last rod A_nA_{n+1} is suspended, also restricted to oscillate about a line parallel to OA_1, and whose physical properties are the same as those of all the other rods. It was shown in [2] that as long as the angular velocity satisfies the condition

$$\omega^2 < -\frac{g}{\alpha(n)} \qquad (0.1)$$

appropriate initial conditions can be chosen so that this system of penduli can undergo small periodic vibrations about the vertical position. So long as (0.1) is satisfied, this vertical position will be an equilibrium position of relative stability. Here α_n is the lowest root of the equation

$$\begin{vmatrix} c_{11} & 2(n-2)+1 & \ldots & 2(n-n)+1 \\ 2(n-2)+1 & c_{22} & \ldots & 2(n-n)+1 \\ \ldots & \ldots & \ldots & \ldots \\ 2(n-n)+1 & 2(n-n)+1 & \ldots & c_{nn} \end{vmatrix} = 0 \qquad (0.2)$$

where

$$c_{\nu\nu} = \frac{2}{3}[3n-(3\nu-1)] + \frac{2n-(2\nu-1)}{2a}x, \qquad 1, \ldots, n$$

Reprint Order No. PMM 14.

It was shown that the roots of equation (0.2) are not only negative, but are also simple. If we denote by θ_ν ($\nu = 1, \ldots, n$) the angles the rods form with the z-axis, and if we write $\theta_\nu = \lambda\, \psi_\nu$, then the initial conditions for which the small periodic vibrations were obtained are of the form

$$\psi_1(0) = \ldots = \psi_n(0) = 0, \qquad \psi_1'(0) = \lambda_{11} + \beta_1, \ldots, \quad \psi_n'(0) = \lambda_{1n} + \beta_n \tag{0.3}$$

Here the numbers $\lambda_{11}, \ldots, \lambda_{1n}$ are given by the set of equations

$$2\left(\tfrac{1}{3} + n - \nu\right)\lambda_\nu \rho^2 + [2(n-\nu)+1]\,\rho^2 \sum_{k=1}^{\nu-1} \lambda_k + \sum_{k=\nu+1}^{n} [2(n-k)+1]\,\rho^2 \lambda_k =$$

$$= 2\omega^2\left(n - \nu + \tfrac{1}{3}\right)\lambda_\nu + \omega^2 \sum_{k=\nu+1}^{n} [2(n-k)+1]\,\lambda_k + \omega^2 [2(n-\nu)+1] \sum_{k=1}^{\nu-1} \lambda_k -$$

$$- \frac{g}{2a}[2(n-\nu)+1]\,\lambda_\nu, \qquad \nu = 1, 2, \ldots n, \qquad \sum_{1}^{0} = \sum_{n+1}^{n} = 0 \tag{0.4}$$

where $\rho^2 = \omega^2 + g/x_1$, and x_1 is the largest root of equation (0.2).

If, however, one chooses initial conditions which do not depend on the largest root of (0.2), the method suggested in the author's previous paper [2] cannot be always used to solve the problem. Let us consider the case of two penduli. Let x_2 be the smaller root of equation (0.2) with $n = 2$. If equation (0.4) is used to obtain λ_{21} and λ_{22} with $n = 2$ and with $\rho^2 = \omega^2 + g/x_2$, and if the initial conditions are chosen as $\psi_1(0) = \psi_2(0) = 0$, $\psi_1'(0) = \lambda_{21} + \beta_1$, and $\psi_2'(0) = \lambda_{22} + \beta_2$, then the previously proposed method will lead to a solution only on the assumption that

$$\sin \frac{k_1 \pi}{k_2} \neq 0, \qquad k_i = \sqrt{-(\omega^2 + g/x_i)}, \qquad i = 1,2 \tag{0.5}$$

It will be shown below that the condition given by (0.5) is not always fulfilled. We shall call this a special case.

In the present paper we show that with the appropriate initial conditions the problem of the existence of small periodic vibrations has a solution also in the above-mentioned special case. Bradistilov [3] has treated the case of two physical penduli without rotation.

1. We shall first show that the special case may actually occur. The roots of equation (0.2), with $n = 2$, are $x_{1,2} = (-14 \pm 4\sqrt{7})\,a/9$. Equation (0.1) for ω^2 can be written, when $n = 2$, in the form

$$\omega^2 < 3g(\sqrt{7} - 2)/2\sqrt{7}\,a \tag{1.1}$$

On the other hand,

$$\frac{k_1}{k_2} = \sqrt{\frac{3g(\sqrt{7}+2) - 2a\sqrt{7}\,\omega^2}{3g(\sqrt{7}-2) - 2a\sqrt{7}\,\omega^2}}$$

It is easily shown that this ratio cannot take on the integral values 0, 1, or 2. If, however, k is an integer greater than or equal to 3, and if we choose

$$\omega^2 = \frac{3g}{2a}\left[1 - \frac{2(k^2+1)}{\sqrt{7}\,(k^2-1)}\right] \tag{1.2}$$

then $k_1/k_2 = k$, which is the special case. We note that (1.2) does not contradict (1.1). The set of differential equations becomes

$$\psi_i'' = f_i(\psi_1, \psi_2, \psi_1', \psi_2', \lambda) \qquad (i = 1, 2) \tag{1.3}$$

where $f_i(\psi_1, \psi_2, \psi_1', \psi_2', \lambda)$ are completely defined functions. In the case $\lambda = 0$, equations (1.3) reduce to the simplified set of equations

$$\begin{aligned} \frac{16}{3}\psi_1'' + 2\psi_2'' - \left(\frac{16}{3}\omega^2 - 3\frac{g}{a}\right)\psi_1 - 2\omega^2\psi_2 = 0 \\ 2\psi_1'' + \frac{4}{3}\psi_2'' - \left(\frac{4}{3}\omega^2 - \frac{g}{a}\right)\psi_2 - 2\omega^2\psi_1 = 0 \end{aligned} \tag{1.4}$$

Let us consider the particular solution of (1.4) with the initial conditions

$$\psi_1(0) = \psi_2(0) = 0, \qquad \psi_1'(0) = \lambda_{21} + k\lambda_{11}N, \qquad \psi_2'(0) = \lambda_{22} + k\lambda_{12}N$$

The numbers λ_{i1} and λ_{i2} (where $i = 1, 2$) are given by equations (0.4) with $n = 2$ and with $\rho^2 = \omega^2 + g/x_i$ (with $i = 1, 2$). This solution is

$$\psi_{i0}(t) = \frac{\lambda_{1i}N}{k_2}\sin kk_2t + \frac{\lambda_{2i}}{k_2}\sin k_2t$$

$$\psi_{i0}'(t) = \lambda_{1i}N\;k\cos\;kk_2t + \lambda_{2i}\cos k_2t \qquad (i = 1,\ 2) \tag{1.5}$$

The solution given by (1.5) is obviously periodic with period $2\pi/k_2$. We shall now show that it is possible to choose β, δ, and N so that the solution of (1.3) with initial conditions

$$\psi_1(0) = \psi_2(0) = 0, \qquad \psi_1'(0) = \lambda_{21} + k\lambda_{11}(N+\beta), \qquad \psi_2'(0) = \lambda_{22} + k\lambda_{12}(N+\beta) \tag{1.6}$$

be periodic with period $2(\pi + \delta)/k_2$. According to a fundamental theorem of Poincaré, the solution of (1.3) with initial conditions (1.6) can be written as

$$\begin{aligned} \psi_i(t, \beta, \lambda) = \psi_{i0}(t) + P_i(t)\beta + \lambda^2[Q_i(t) + R_i(t)\beta + \cdots] \\ \psi_i'(t, \beta, \lambda) = \psi_{i0}'(t) + P_i'(t)\beta + \lambda^2[Q_i'(t) + R_i'(t)\beta + \cdots] \end{aligned} \qquad (i = 1,\ 2) \tag{1.7}$$

Inserting (1.7) into (1.3) and equating the coefficients of β, λ^2, and $\lambda^2\beta$, we obtain a set of differential equations for the functions $P_i(t)$, $Q_i(t)$, and $R_i(t)$. Here we make use also of the initial conditions (1.6). We then calculate $Q_i(\pi/k_2)$, and $R_i(\pi/k_2)$.

2. The differential equations of motion have the property that if $\psi_i(t)$ is a solution, then so is $-\psi_i(2q - t)$. Therefore [1], if the conditions

$$\psi_i(q) = 0 \qquad (i = 1,\ 2) \tag{2.1}$$

are satisfied, one may assert that there exists a periodic solution with period $2q$. In our case $q = (\pi + \delta)/k_2$, and therefore conditions (2.1) become

$$\psi_i\left(\frac{\pi+\delta}{k_2},\ \beta,\ \lambda\right) = 0 \qquad (i = 1,\ 2) \tag{2.2}$$

or, with (1.7),

$$-\frac{\lambda_{2i}}{k_2}\sin\delta + \frac{\varepsilon_2 N}{k_2}\lambda_{1i}\sin k\delta + \frac{\varepsilon_2\lambda_{1i}}{k_2}\sin k\delta\beta + \\ + \lambda^2\left[Q_i\left(\frac{\pi+\delta}{k_2}\right) + R_i\left(\frac{\pi+\delta}{k_2}\right)\beta + \cdots\right] = 0 \tag{2.3}$$

Here ϵ_2 is +1 or −1 depending on whether k is even or odd. Expanding (2.3) in a power series in δ, we obtain

$$\delta\left[-\frac{\lambda_{21}}{k_2} + \frac{\varepsilon_2 N}{k_2}\lambda_{1i}k + \frac{\varepsilon_2\lambda_{1i}}{k_2}k\beta + \cdots\right] + \\ + \lambda^2\left[Q_i\left(\frac{\pi}{k_2}\right) + R_i\left(\frac{\pi}{k_2}\right)\beta + \cdots\right] = 0 \quad (i = 1,2) \tag{2.4}$$

This set of equations will be satisfied by values of δ and λ^2 in the neighborhood of the point $\delta = \beta = \lambda = 0$, if the equation

$$\left(-\frac{\lambda_{21}}{k_2} + \frac{\varepsilon_2 N}{k_2}\lambda_{11}k + \frac{\varepsilon_2}{k_2}\lambda_{11}k\beta + \cdots\right)\left[Q_2\left(\frac{\pi}{k_2}\right) + R_2\left(\frac{\pi}{k_2}\right)\beta + \cdots\right] - \\ -\left(-\frac{\lambda_{22}}{k_2} + \frac{\varepsilon_2 N}{k_2}\lambda_{12}k + \frac{\varepsilon_2}{k_2}\lambda_{12}k\beta + \cdots\right)\left[Q_1\left(\frac{\pi}{k_2}\right) + R_1\left(\frac{\pi}{k_2}\right)\beta + \cdots\right] = 0 \tag{2.5}$$

is also satisfied.

Equation (2.5) can be used to determine β if the conditions

$$\left(-\frac{\lambda_{21}}{k_2} + \frac{\varepsilon_2 N}{k_2}\lambda_{11}k\right)Q_2\left(\frac{\pi}{k_2}\right) - \left(-\frac{\lambda_{22}}{k_2} + \varepsilon_2\frac{N}{k_2}\lambda_{12}k\right)Q_1\left(\frac{\pi}{k_2}\right) = 0 \tag{2.6}$$

$$\left(-\frac{\lambda_{21}}{k_2} + \frac{\varepsilon_2 N}{k_2}\lambda_{11}k\right)R_2\left(\frac{\pi}{k_2}\right) - \left(-\frac{\lambda_{22}}{k_2} + \frac{\varepsilon_2 N}{k_2}\lambda_{12}k\right)R_1\left(\frac{\pi}{k_2}\right) + \\ + \frac{\varepsilon_2\lambda_{11}}{k_2}kQ_2\left(\frac{\pi}{k_2}\right) - \frac{\varepsilon_2\lambda_{12}}{k_2}kQ_1\left(\frac{\pi}{k_2}\right) \neq 0 \tag{2.7}$$

are satisfied.

We can determine the parameter N from (2.6). If $k > 3$, calculations show that (2.6) is an equation of third degree with respect to N. In addition, it is found that one of the roots of this equation vanishes, and that the other two are determined by the equation

$$x(k^2)N^2 + y(k^2) = 0 \tag{2.8}$$

where

$$x(k^2) = 18(1568 - 571\sqrt{7})k^4 + (22946 + 19691\sqrt{7})k^2 + 9(259\sqrt{7} + 910)$$

$$y(k^2) = 9(910-259\sqrt{7})k^4 + (22946-19691\sqrt{7})k^2 + 18(571\sqrt{7} + 1568)$$

Obviously $x(k^2) > 0$. Since k is an integer greater than 3, $k^2 \geqslant 16$. Direct verification shows that $y(k^2 = 16) > 0$. On the other hand $y(k^2)$ is a quadratic function of k^2 and therefore increases monotonically for

$$k^2 > \frac{19691\sqrt{7} - 22946}{18(910-259\sqrt{7})}$$

The number on the right-hand side, however, is less than 16. Therefore $y(k^2) > 0$, which means that the only real root of (2.8) is $N = 0$.

Condition (2.7) is obtained from (2.6) by differentiating with respect to the parameter N, which leads to

$$3x(k^2)N^2 + y(k^2) \neq 0 \tag{2.9}$$

If $N = 0$, the latter condition is satisfied, since $y(k^2) > 0$. Thus the results obtained above can be formulated as follows.

Let us assume that we are dealing with the special case in which k is an integer greater than 3 and

$$\omega^2 = \frac{3g}{2a}\left[1 - \frac{2(k^2+1)}{\sqrt{7}(k^2-1)}\right]$$

It is possible to choose β and δ as functions of λ in the neighborhood of the point $\beta = \delta = \lambda = 0$, so that the motion of the system with initial conditions $\psi_1(0) = \psi_2(0) = 0$, $\psi_1'(0) = \lambda_{21} + k\lambda_{11}\beta$, and $\psi_2'(0) = \lambda_{22} + k\lambda_{12}\beta$ be periodic with period $2(\pi + \delta)k_2$. As a first approximation we obtain the motion

$$\psi_i(t) = \frac{\lambda_{2i}}{k_2}\sin k_2 t, \qquad \psi_i'(t) = \lambda_{2i}\cos k_2 t \qquad (i = 1, 2)$$

Let us consider the case $k = 3$. Here condition (2.6) becomes

$$(\lambda_{21} + 3\lambda_{11}N)Q_2\left(\frac{\pi}{k_2}\right) - (\lambda_{22} + 3\lambda_{12}N)Q_1\left(\frac{\pi}{k_2}\right) = 0 \tag{2.10}$$

By calculation we arrive at

$$81(11.578-3023\sqrt{7})N^3 + 2187(47\sqrt{7} - 119)N^2 -$$
$$- 2187(61\sqrt{7} - 154)N - (1351+1217\sqrt{7}) = 0 \tag{2.11}$$

Let us denote the left-hand side of (2.11) by the symbol $z(N)$. Direct verification will show that $\dot{z}(-1/6) > 0$, $\dot{z}(0) < 0$, and $\dot{z}(+\infty) > 0$. It follows from this that the equation $z(N) = 0$ has roots N_1 and N_2 in the intervals $-1/6 < N_1 < 0$, and $N_2 > 0$. In addition, if $-1/6 \leqslant N \leqslant 0$, then

$$z(N) = 2187(47\sqrt{7} - 119)(-N)^2 + 2187(61\sqrt{7} - 154)(-N) - 81(11578 -$$
$$- 3023\sqrt{7})(-N)^3 - (1351 + 1217\sqrt{7}) \leqslant 2187(47\sqrt{7} - 119)\frac{1}{36} +$$
$$+ 729(61\sqrt{7} - 154)\frac{1}{2} - (1351 + 1217\sqrt{7}) < 0$$

It follows that (2.11) has only one real root, which is positive and simple. Thus condition (2.7), which here takes on the form $\dot{z}(N) \neq 0$, is satisfied. We may therefore formulate the following result.

If $k = 3$ and $\omega^2 = 3(14 - 5\sqrt{7})\, g/28a$, it is possible to choose β and δ as functions of λ in the neighborhood of the point $\beta = \delta = \lambda = 0$ so that with the initial conditions

$$\psi_1(0) = \psi_2(0) = 0, \qquad \psi_1'(0) = \lambda_{21} + 3\lambda_{11}(N + \beta), \qquad \psi_2'(0) = \lambda_{22} + 3\lambda_{12}(N + \beta)$$

the motion will be periodic with period $2(\pi + \delta)/k_2$. Here N is the only root of (2.11).

The first approximation gives

$$\psi_i(t) = \frac{1}{k_2}(\lambda_{1i} N \sin 3k_2 t + \lambda_{2i} \sin k_2 t) \qquad (i = 1, 2)$$
$$\psi_i'(t) = 3\lambda_{1i} N \cos 3k_2 t + \lambda_{2i} \cos k_2 t$$

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Translated by E.J.S.